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Second Prize at the ICM 2006 poster competition

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Icosian Game (1859)

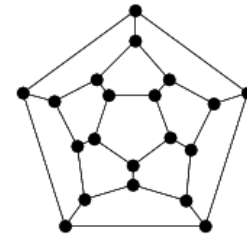


Icosian Game

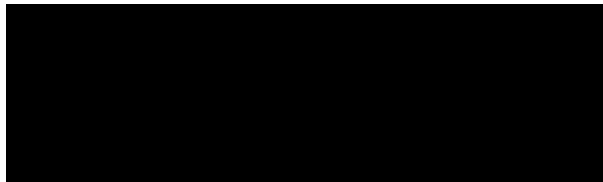
W. R. Hamilton

hamiltonian cycle (path) $C_{|G|} \subset G$ ($P_{|G|} \subset G$) in a dodecahedral graph

N. L. Biggs, E. K. Lloyd, R. J. Wilson, Graph Theory 1736-1936, Clarendon Press, Oxford, 1986.



Chinese Rings and Gray Code



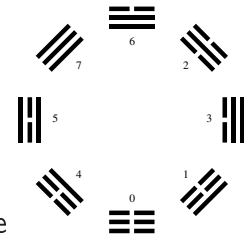
Chinese Rings

modeled by state graph $B^n := \{0, 1\}^n \sim P_{2^n}$

leads to Gros sequence 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, ... and Gray code

hamiltonian cycle on n -cube

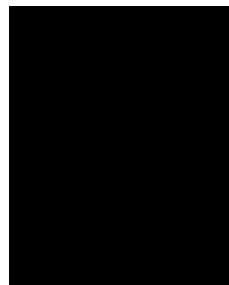
S. N. Afriat, The Ring of Linked Rings, Duckworth, London, 1982.



Tower of Hanoi (1883)



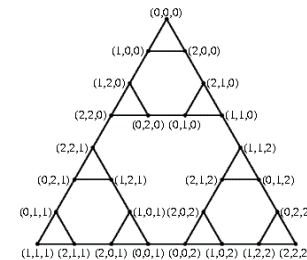
Title plate



F. É. A. Lucas

A. M. Hinz, The Tower of Hanoi, Enseign. Math. (2) 35 (1989), 289-321.

A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, European J. Combin. 26 (2005), 693-708.



modeled by state graph $T^n := \{0, 1, 2\}^n$

connected, planar, hamiltonian, $\text{diam}(T^n) = 2^n - 1$, $\chi(T^n) = 3 = \chi'(T^n)$

at most two shortest paths between any two vertices
 Wiener index $W(T^n) = \frac{233}{885} 18^n - \frac{1}{6} 9^n + \frac{3}{59} \left(2 + \frac{3}{17} \sqrt{17}\right) \left(\frac{1}{2} (5 + \sqrt{17})\right)^n - \frac{3}{10} 3^n + \frac{3}{59} \left(2 - \frac{3}{17} \sqrt{17}\right) \left(\frac{1}{2} (5 - \sqrt{17})\right)^n$



Tower of Hanoi platform



Tower of London platform

T. Shallice, Specific impairments of planning, Philos. Trans. Roy. Soc. London B 298 (1982), 199–209.

A. M. Hinz et al., A Mathematical Model and Computer Tool for the Tower of Hanoi and Tower of London Tasks, in preparation, 2007.



Tower of London modeled by state graph L_3^3 : connected, planar, non-hamiltonian, $\text{diam}(L_3^3) = 8$ up to eight shortest paths between two vertices

subgraphs of Oxford graphs O_3^n : connected, non-planar ($n \geq 3$) digraph for Lucas's Second Problem (1883)

Comparison of Tower Graphs

T^n, O_3^n, L_3^3 with its variants $L_3^2 \subset O_3^2, \tilde{L}_3^3 \subset O_3^3, \tilde{L}_4^4 \subset O_3^4, \tilde{L}_5^5 \subset O_3^5$, and the 4-pegs versions $Q^n := \{0, 1, 2, 3\}^n, O_4^n, L_4^4 \subset O_4^4$, and $L_5^5 \subset O_4^5$ have the following quantitative properties:

G	$ G $	$\ G\ $	δ	$\overline{\text{deg}}$	Δ	\bar{d}	$\text{diam}(G)$	equi-sets (number)	equi-sets (sizes)
T^2	9	12	2	2.667	3	1.778	3	12	12×6
T^3	27	39	2	2.889	3	3.893	7	117	117×6
T^4	81	120	2	2.963	3	8.102	15	1080	1080×6
T^5	243	363	2	2.988	3	16.523	31	9801	9801×6
T^6	729	1092	2	2.996	3	33.370	63	88452	88452×6
$O_{3,2}^2$	12	18	2	3.000	4	1.958	4	12	$10 \times 12 + 2 \times 6$
$O_{3,3}^3$	60	108	2	3.600	6	3.647	7	102	$95 \times 36 + 6 \times 18 + 1 \times 12$
$O_{3,4}^4$	360	720	2	4.000	6	5.596	10	914	$881 \times 144 + 33 \times 72$
$O_{3,5}^5$	2520	5400	2	4.286	6	7.738	13	8885	$8748 \times 720 + 137 \times 360$
$L_{3,2}^2$	10	14	2	2.800	4	1.860	4	23	$22 \times 4 + 1 \times 2$
$L_{3,3}^3$	36	54	2	3.000	4	4.236	8	210	210×6
$L_{3,4}^4$	54	96	2	3.778	6	3.562	7	239	$238 \times 12 + 1 \times 6$
$L_{3,5}^5$	264	504	2	3.818	6	5.828	10	2893	2893×24
$L_{3,6}^6$	1320	2520	2	3.818	6	8.868	14	14509	14509×120
Q_3^3	16	36	3	4.500	5	1.781	3	13	$7 \times 24 + 6 \times 12$
Q_3^4	64	168	3	5.250	6	3.035	5	182	$154 \times 24 + 28 \times 12$
Q_3^5	256	720	3	5.625	6	4.666	9	2780	$2660 \times 24 + 120 \times 12$
Q_3^6	1024	2976	3	5.813	6	6.733	13	43896	$43400 \times 24 + 496 \times 12$
$O_{4,2}^2$	20	48	3	4.800	6	1.910	4	13	$4 \times 48 + 7 \times 24 + 1 \times 12 + 1 \times 8$
$O_{4,3}^3$	120	360	3	6.000	9	3.290	6	126	$76 \times 144 + 44 \times 72 + 1 \times 48 + 5 \times 24$
$O_{4,4}^4$	840	2880	3	6.857	12	4.815	9	1400	$1068 \times 576 + 298 \times 288 + 1 \times 192 + 9 \times 144 + 23 \times 96 + 1 \times 72$
$O_{4,5}^5$	6720	25200	3	7.500	12	6.452	12	16852	$14584 \times 2880 + 2146 \times 1440 + 2 \times 960 + 120 \times 480$
$L_{4,2}^2$	480	1464	3	6.100	9	4.765	9	9580	9580×24
$L_{4,3}^3$	2640	8280	4	6.273	9	6.819	12	58058	58058×120

Higher Base Hanoi Graphs

presumed minimal solution for perfect to perfect problem in Q^n
 Frame-Stewart-Conjecture: $pms = d(p, \tilde{p})$; true for $n \leq 30$.

eccentricity $\varepsilon(p) = d(p, \tilde{p})$ for $n < 15$, but:

n	$d(p, \tilde{p})$	$\varepsilon(p)$
15	129	130
20	289	294
21	321	341
22	385	394

$\text{diam}(Q^n) = d(p, \tilde{p})$ for $n \leq 12$; **open**: $\text{diam}(Q^n) = \varepsilon(p)$?

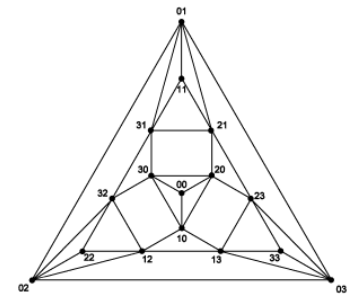
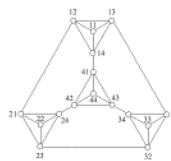
$H_p^n \subset O_p^n$ Hanoi/Oxford graphs of base p and exponent n ; **open**: diameters

$$\chi(H_p^n) = p, \chi'(H_p^n) = \Delta(H_p^n) = \binom{p}{2} - \binom{p-n}{2} \cdot (n \leq p-2) \text{ for } p, n > 1$$

A.M.Hinz & D.Parisse (2006)

H_p^n hamiltonian; planar only for $p < 4$ and $p = 4, n = 1, 2$

connections to Sierpiński graphs S_p^n : $S_2^n \sim B^n, S_3^n \sim T^n$, but S_p^n and H_p^n are not isomorphic for $p > 3, n > 1$.



J.-B. Bode, A. M. Hinz, Results and open problems on the Tower of Hanoi, Congr. Numer. 139 (1999), 113–122.

R. E. Korf, A. Felner, Heuristic Search for Multiple Goal States, Preprint, 2006.

A. M. Hinz, D. Parisse, On the Planarity of Hanoi Graphs, Exposition. Math. 20 (2002), 263–268.

S. Klavžar, U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47(122) (1997), 95–104.